Winter School in Abstract Analysis 2023

Infinitely ludic categories

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Joint work with P. Szeptycki and W. Tholen

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Intuitively, we're interested in games that:

- are between two players (ALICE and BOB);
- are "turn-based" (ALICE starts);
- two players compete;
- with no draws;
- of "perfect information";
- infinite (countable) runs.

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Definition

An *infinite game* is a pair $\mathrm{G} = (T, A)$ with $T \subset M^{<\omega}$ and $A \subset M^{\omega}$ for some set M such that

(I) If $t \in T$, then $t \restriction k \in T$ for all $k \leq |t|$;

(II) For all $t \in T$ there is an $x \in M$ such that $t^\frown x \in T$;

(III) $A \subset \mathrm{Runs}(G) = \{R \in M^{\omega} : R \restriction n \in T \text{ for all } n \in \omega\}$.

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If $t = \langle x_0, \ldots, x_{n-1} \rangle$, then

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If $t = \langle x_0, \ldots, x_n \rangle$ e $k \leq n$, then

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An *infinite game* is a pair $\mathrm{G} = (T, A)$ with $T \subset M^{<\omega}$ and $A \subset M^{\omega}$ for some set M such that

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All of our games will be infinite in this talk, so we will omit the word "infinite" from now on.

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- If $t \in T$ and $|t|$ is even, then we say that it is $\textsc{Alice}$'s *turn* and $\left\{ x \in M : t^\frown x \in T \right\}$ is the set of all valid choices that $\textsc{Alice}$ can make at t.

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- If $t \in T$ and $|t|$ is even, then we say that it is Alice's *turn* and $\{ x \in M : t^\frown x \in T \}$ is the set of all valid choices that Alice can make at t.
- If $t \in T$ and $|t|$ is odd, then we say that it is Bob's *turn* and $\{ x \in M : t^\frown x \in T \}$ is the set of all valid choices that Bob can make at t.
- If $t \in T$ and $|t| = 2n$ or $|t| = 2n + 1$, then we say that t is at the *nth inning*.

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- If $t \in T$ and $|t|$ is even, then we say that it is ALICE's *turn* and $\{ x \in M : t^\frown x \in T \}$ is the set of all valid choices that ALICE can make at t.
- If $t \in T$ and $|t|$ is odd, then we say that it is BOB's *turn* and $\{ x \in M : t^\frown x \in T \}$ is the set of all valid choices that BOB can make at t.
- If $t \in T$ and $|t| = 2n$ or $|t| = 2n + 1$, then we say that t is at the *nth inning*.
- We say that A is the *payoff set* of G: a run R is won by ALICE if $R \in A$ (and won by BOB otherwise).

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  - ALICE chooses a non-empty open set U_n contained in the open set V_{n-1} chosen by BOB in the previous inning;

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  - ALICE chooses a non-empty open set U_n contained in the open set V_{n-1} chosen by BOB in the previous inning;
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  - ALICE chooses a non-empty open set U_n contained in the open set V_{n-1} chosen by BOB in the previous inning;
  - BOB responds with a non-empty open set $V_n \subset U_n$.

Then BOB wins the run $\langle U_0, V_0, \ldots, U_n, V_n, \ldots \rangle$ if $\bigcap_{n \in \omega} V_n \neq \emptyset$ (and ALICE wins otherwise).

Example (World's most boring game)

$$(\emptyset, \emptyset)$$

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(by vacuity, with $M = \emptyset$)

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A strategy for ALICE in the game $G = (T, A)$ is a subgame given by $\gamma \subset T$ which satisfies the following conditions:

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(b) if $s \in \gamma$ ALICE's turn, then there is a unique x such that $s^\frown x \in \gamma$;

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A strategy for Alice in the game $G = (T, A)$ is a subgame given by $\gamma \subset T$ which satisfies the following conditions:

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If Alice wins every run of the subgame given by γ, then we say that γ is a *winning* strategy for Alice.

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We denote the claim "there is a winning strategy for Alice in G" by Alice$\uparrow G$ (and Alice$\not\uparrow G$ as its negation).

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Definition (for Bob)

A strategy for Bob in the game $G = (T, A)$ is a subgame given by $\sigma \subset T$ which satisfies the following conditions:

(a) $\sigma \neq \emptyset$;

(b) if $s \in \sigma$ is Bob's turn, then there is a unique $x \in M$ such that $s^\frown x \in \sigma$;

(c) if $s \in \sigma$ is Alice's turn, then $s^\frown x \in \sigma$ for all $x \in M$ such that $s^\frown x \in T$.

If Bob wins every run of the subgame given by γ, then we say that γ is a *winning* strategy for Bob.
We denote the claim "there is a winning strategy for Bob in G" by Bob $\uparrow G$ (and Bob $\not\uparrow G$ as its negation).

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An A-morphism $G_1 \xrightarrow{f} G_2$ between the games $G_1 = (T_1, A_1)$ and $G_2 = (T_2, A_2)$ is a mapping $f\colon T_1 \to T_2$ such that:

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(c) For every run $R \in A_1$ in the game G_1, $\bigcup f[R] \in A_2$.

Definition ($\mathbb{B}$-morphism)

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(a) For all $t \in T_1$, $|f(t)| = |t|$;

(b) For every $t \in T_1$ and $k \leq |t|$, $f(t \upharpoonright k) = f(t) \upharpoonright k$;

(c) For every run $R \notin A_1$ in the game G_1, $\bigcup f[R] \notin A_2$.

Examples

- For every game $G = (T, A)$, $\mathrm{id}\colon T \to T$ is an A-morphism (and also a B-morphism) from G into G.

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- If $G' = (T', A')$ is a subgame of $G = (T, A)$, then the inclusion $i\colon T' \to T$ is an A-morphism (and also a B-morphism).

Proposition

Let $G_1 = (T_1, A_1)$, $G_2 = (T_2, A_2)$ and $G_3 = (T_3, A_3)$ be games. If $f\colon T_1 \to T_2$ is an A-morphism from G_1 into G_2 and $g\colon T_2 \to T_3$ is an A-morphism from G_2 into G_3, then $g \circ f$ is an A-morphism from G_1 into G_3.

Proposition

Let $G_1 = (T_1, A_1)$, $G_2 = (T_2, A_2)$ and $G_3 = (T_3, A_3)$ be games. If $f : T_1 \to T_2$ is a B-morphism from G_1 into G_2 and $g : T_2 \to T_3$ is a B-morphism from G_2 into G_3, then $g \circ f$ is a B-morphism from G_1 into G_3.

Let us recall the following theorem from Group Theory, which states that symmetric groups are, in some sense, “universal”:

Theorem (A. Cayley – 1854)

For every group G there is a set $X(G)$ such that G is isomorphic to a subgroup of the symmetric group of $X(G)$.

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We also have the "universality" of the Banach-Mazur game:

Theorem (D., P. Szeptycki, W. Tholen – 202?)

For every game G there is a metrizable space $K(G)$ such that G is isomorphic to a subgame of the Banach-Mazur game over $K(G)$.

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- **Games**$_{\mathrm{A}}$: objects are games and morphisms are A-morphisms.
- **Games**$_{\mathrm{B}}$: objects are games and morphisms are B-morphisms.

Proposition

The categories $\mathbf{Games}_{\mathrm{A}}$ *and* $\mathbf{Games}_{\mathrm{B}}$ *are isomorphic.*

Theorem (D., P. Szeptycki, W. Tholen – 202?)

Suppose **C** *is either* $\mathbf{Games}_{\mathrm{A}}$ *or* $\mathbf{Games}_{\mathrm{B}}$*. Then:*

Theorem (D., P. Szeptycki, W. Tholen – 202?)

Suppose **C** *is either* $\textbf{Games}_{\mathrm{A}}$ *or* $\textbf{Games}_{\mathrm{B}}$*. Then:*

- **C** *is complete and co-complete.*

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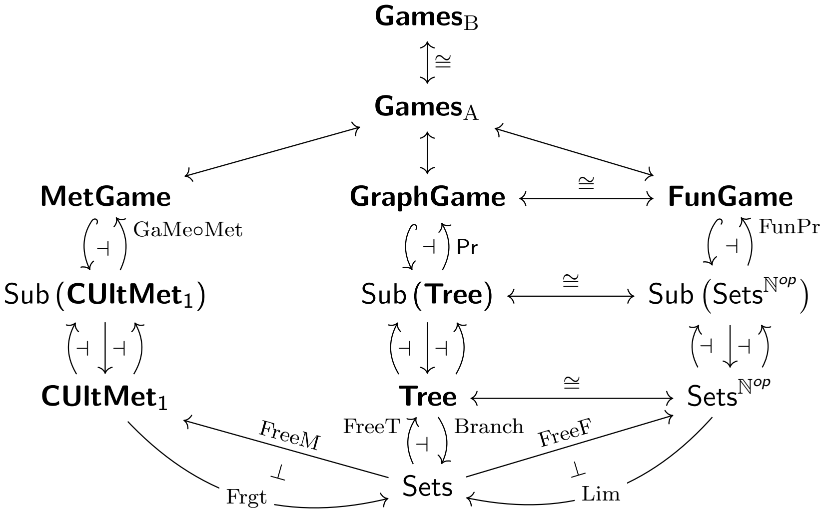
Suppose **C** *is either* $\mathbf{Games}_{\mathrm{A}}$ *or* $\mathbf{Games}_{\mathrm{B}}$*. Then:*

- **C** *is complete and co-complete.*
- **C** *is cartesian closed.*
- **C** *has orthogonal factorization systems.*


Games_B
≅
Games_A
MetGame
GraphGame
≅
FunGame
⊣
GaMe∘Met
⊣
Pr
⊣
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Sub (CUltMet_1)
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≅
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⊣
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CUltMet_1
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Sets
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Topological games as functors

Topological games as functors

Example ($G_1(\Omega_x, \Omega_x)$)

Given a non-empty topological space X and a fixed $x \in X$, consider the following game: in each inning $n \in \omega$,

- Alice chooses $A_n \subset X$ such that $x \in \overline{A_n}$;
- Bob responds with $a_n \in A_n$.

Bob wins the run $\langle A_0, a_0, \ldots, A_n, a_n, \ldots \rangle$ if, for every $k \in \mathbb{N}$, $x \in \overline{\{ a_n : n \geq k \}}$ (Alice wins otherwise).

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The game $G_1(\Omega_x, \Omega_x)$ can naturally be seen as a covariant functor from Top_* into $\mathbf{Games}_{\mathrm{B}}$:

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$$\mathrm{Tight}((X,x)) \xrightarrow{\mathrm{Tight}(f)} \mathrm{Tight}((Y,y))$$

$$\langle A_0, a_0, \ldots, A_n, a_n \rangle \longmapsto \langle f[A_0], f(a_0), \ldots, f[A_n], f(a_n) \rangle$$

Topological games as functors

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Example ($G_1(\Omega, \Omega)$)

Given a topological space X, consider the following game: in each inning $n \in \omega$,

- Alice chooses an ω-cover $\mathcal{U}_n$, that is, an open cover $\mathcal{U}_n$ such that
$$\forall F \in [X]^{<\omega} \exists U \in \mathcal{U}_n (F \subset U),$$
- Bob responds with $U_n \in \mathcal{U}_n$.

Bob wins the run $\langle \mathcal{U}_0, U_0, \ldots, \mathcal{U}_n, U_n, \ldots \rangle$ if, for every $k \in \omega$, $\{ U_n : n \geq k \}$ is an ω-cover (Alice wins otherwise).

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- On morphisms, given a continuous $f : X \to Y$, let

$$\mathrm{Cover}(Y) \xrightarrow{\mathrm{Cover}(f)} \mathrm{Cover}(X)$$

$$\langle \mathcal{U}_0, U_0, \ldots, \mathcal{U}_n, U_n \rangle \longmapsto \langle f^{-1}[\mathcal{U}_0], f^{-1}(U_0), \ldots, f^{-1}[\mathcal{U}_n], f^{-1}(U_n) \rangle$$

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Theorem

If X is a $T_{3\frac{1}{2}}$ space, then

- $\mathrm{A}\uparrow \mathsf{G}_1(\Omega, \Omega)$ *over* $X \iff \mathrm{A}\uparrow \mathsf{G}_1(\Omega_{\bar{0}}, \Omega_{\bar{0}})$ *over* $\mathrm{C_p}(X)$ *(*M. Scheepers – 1997*)*
- $\mathrm{B}\uparrow \mathsf{G}_1(\Omega, \Omega)$ *over* $X \iff \mathrm{B}\uparrow \mathsf{G}_1(\Omega_{\bar{0}}, \Omega_{\bar{0}})$ *over* $\mathrm{C_p}(X)$ *(*M. Scheepers – 2014*)*

Theorem

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It would be neat to find some natural transformations that entail the above result.

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Issues:

Theorem

If X is a $T_{3\frac{1}{2}}$ space, then

- $\mathrm{A}\uparrow \mathsf{G}_1(\Omega,\Omega)$ *over* $X \iff \mathrm{A}\uparrow \mathsf{G}_1(\Omega_{\bar{0}},\Omega_{\bar{0}})$ *over* $\mathrm{C_p}(X)$ *(*M. Scheepers – 1997*)*
- $\mathrm{B}\uparrow \mathsf{G}_1(\Omega,\Omega)$ *over* $X \iff \mathrm{B}\uparrow \mathsf{G}_1(\Omega_{\bar{0}},\Omega_{\bar{0}})$ *over* $\mathrm{C_p}(X)$ *(*M. Scheepers – 2014*)*

It would be neat to find some natural transformations that entail the above result.

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- The domain of $\mathsf{G}_1(\Omega_x, \Omega_x)$'s functor is Top_*, while the domain of $\mathsf{G}_1(\Omega, \Omega)$'s functor is Top;
- $\mathsf{G}_1(\Omega_x, \Omega_x)$'s functor is covariant, while $\mathsf{G}_1(\Omega, \Omega)$'s functor is contravariant.

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- on objects, $\mathrm{D}(V) = V^*$,
- on morphisms, if $f : V_1 \to V_2$ is a linear map,

$$\begin{aligned} V_2^* &\xrightarrow{\mathrm{D}(f)} V_1^* \\ \varphi_2 &\longmapsto \varphi_2 \circ f \end{aligned}$$

Consider the contravariant functor $\mathrm{C_p} : \mathsf{Top} \to \mathsf{Top}_*$ such that

- on objects, $\mathrm{C_p}(X) = (\mathrm{C_p}(X), \bar{0})$,
- on morphisms, if $f : X_1 \to X_2$ is continuous,

$$(\mathrm{C_p}(X_2), \bar{0}) \xrightarrow{\mathrm{C_p}(f)} (\mathrm{C_p}(X_1), \bar{0})$$
$$\varphi_2 \longmapsto \varphi_2 \circ f$$

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That is, we want to find some "nice" natural transformations between Cover with Tight $\circ \mathrm{C_p}$.

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Intuitively, “nice” natural transformations are natural transformations that, for each $T_{3\frac{1}{2}}$ space X, somehow “translates” winning strategies in $\mathrm{Tight}(X)$ to winning strategies in $\mathrm{Cover}(X)$ and vice-versa.

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Indeed, we have:

Informal Theorem (D., P. Szeptycki, W. Tholen – 202?)

There are two “nice” transformations that, together, entail Scheepers result.

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- Let $(\mathrm{Tight} \circ \mathrm{C_p}) \xrightarrow{\eta} \mathsf{Cover}$ be such that

$$\begin{array}{ccc} (\mathrm{Tight} \circ \mathrm{C_p})(X) & \xrightarrow{\eta_X} & \mathsf{Cover}(X) \\ \langle A_0, \varphi_0, \ldots, A_n, \varphi_n \rangle & \longmapsto & \langle \mathcal{U}_0(A_0), \varphi_0^{-1}(I_0), \ldots, \mathcal{U}_0(A_n), \varphi_n^{-1}(I_0) \rangle \end{array}$$

  where

$$\mathcal{U}_0(A) = \{ \varphi^{-1}(]-1,1[) : \varphi \in A \} .$$

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- Let $(\mathrm{Tight} \circ \mathrm{C_p})' \xrightarrow{\varepsilon} \mathrm{Cover}'$ be such that

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Děkuji!

Thank you!